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"EXPLORING NEAR EQUIANGULAR FRAMES: PROPERTIES AND RESULTS"

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ABSTRACT
This paper delves into the realm of near equiangular frames, investigating their properties and significance within the context of signal processing, compressed sensing, and related fields. Beginning with an introduction to the topic, the paper proceeds to explore equiangular frames as a foundation. The focus then shifts to the central subject of near equiangular frames, examining their characteristics and applications. Key results pertaining to these frames are presented, shedding light on their utility and potential in various domains. In conclusion, the paper highlights the importance of near equiangular frames and their role in enhancing signal representation and analysis.

KEYWORDS
Equiangular frames, near equiangular frames, signal processing, compressed sensing, frame theory, frame design, frame properties, frame applications.

INTRODUCTION

In recent years, the field of signal processing and related disciplines has witnessed a surge of interest in the study of frames, a fundamental concept with diverse applications in areas such as compressed sensing, communication systems, and data analysis. Among the various types of frames, equiangular frames have garnered particular attention due to their unique properties that offer advantages in signal representation and analysis. Equiangular frames are sets of vectors in a finite-dimensional space characterized by nearly equal angles between any pair of vectors. These frames have found applications in fields like quantum information theory, coding theory, and even in the construction of error-correcting codes.

While equiangular frames present intriguing theoretical possibilities, their perfect construction often proves to be challenging, and in practice, complete equiangularity may be unattainable. This leads us to the concept of near equiangular frames – an area that has emerged as a compromise between theoretical ideals and practical feasibility. Near equiangular frames relax the strict equiangular condition, allowing for slight deviations in the angles between vectors while still preserving the desirable properties of equiangular frames.

In this paper, we delve into the realm of near equiangular frames, aiming to provide a comprehensive exploration of their characteristics, construction, and significance. In the following sections, we will first revisit the fundamentals of equiangular frames to establish a foundational understanding. Subsequently, our focus will shift to the heart of this paper – the concept of near equiangular frames. We will investigate the implications of relaxing the equiangular constraint and delve into the specific properties that make near equiangular frames valuable in various applications.

Furthermore, this paper will present key results and insights concerning near equiangular frames. These results will not only shed light on the mathematical intricacies of these frames but also highlight their potential in practical scenarios. Through comprehensive analysis and discussion, we will demonstrate the relevance of near equiangular frames in enhancing signal representation, reducing redundancy, and improving the efficiency of various signal processing tasks.

In conclusion, our exploration of near equiangular frames underscores their significance as a bridge between theoretical elegance and real-world feasibility. By providing a nuanced understanding of their properties and implications, this paper contributes to the broader frame theory landscape and offers valuable insights for researchers and practitioners seeking to optimize signal processing techniques.

In many applications of mathematics, bounded frames are used. Besides boundedness, while dealing with applications of engineering and sciences, the angle between elements of a frame plays a crucial role. The measure of angle is done with the help of inner product and then the other important aspect of the theory of frame is to consider the inner product and its various consequences in applications. To explain it further, the best and convenient representation of a vector is given by orthonormal bases, which are bounded and perpendicular to each other. Since, it is a basis, so the condition of linear independence is there which itself is a very rigid condition. Similar these convenient representations are given by a particular class of frames. Frames of this class are referred as equiangular frames. These frames plays a significant roles in physical problems like line packing, in graph theory and other similar fields. Uses of equiangular tight frames arise in different application areas namely communications or communication systems, quantum informationprocessing, image processing systems and coding theory. Consequently, construction of equiangular tight frames andthe conditions under which these frames exist has gained a noteworthy attention of researchers recently.

In the present paper , near equiangular frames in Hilbert spaces, which are generalized equiangular frames, are discussed and studied. An introduction of equiangular frames is given in Section 3.2. A brief development of equiangular frames together with some of their applications is provided here for the interest of the readers and for completion. Formal definition and examples of equiangular frames are given. In Section 3.3, generalizing equiangular frames, near equiangular frames are introduced and studied. Some results concerning near equiangular frames are provided and discussed in Section 3.4. Various consequences of near equiangular frames are discussed in the settings of Hilbert spaces. Sufficient number of illustrative examples are provided in support of these results for better understanding. Some applications of these frames are discussed. **Equiangular Frames**

In a Hilbert space, one particular type of bases is very important and convenient for applications. These bases are orthonormal bases. An orthonormal basis is the collection of elements with unit norm and these elements are mutually perpendicular.

In particular, if $\{x_n\}$ is an orthonormal basis, then it satisfies two conditions

1. $\|x_n\| = 1$ i.e each element is of unit norm
2. $\langle x_n, x_m \rangle = 0$ if $n \neq m$ i.e. elements are mutually perpendicular.

In case of frames, due to frame inequality, each element is having finite norm and through a systematic or suitable transformation, the norm of each element can be made unit without actually distorting its structure.

The second condition can be interpreted as the angle between any two vectors of the basis is same. Exploiting this condition, researchers generalized the notion and introduced the notion of equiangular frames.

Definition 1. A frame $\{x_n\}$ is said to be equiangular frame if there exists a scalar k such that

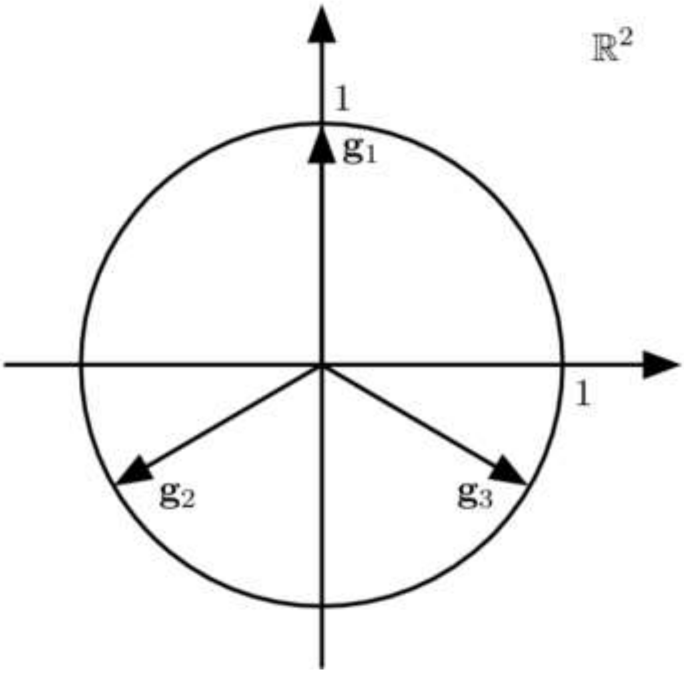
$$|\langle x_n, x_m \rangle| = k; n \neq m.$$

Clearly, if one sets $k = 0$, then the equiangular frames and orthonormal basis or frames and hence equiangular frames are generalization of orthonormal bases.

Few examples of equiangular frames are given below.

Example 1. Mercedes Benz frames in 2-D.

This frame is in 2-Dimensional space with three vectors which have same angle between any pair.
The geometrical representation of the frame is given below



Three vectors g_1, g_2 and g_3 from the frame.
It is evident from the picture that the angle between the vectors is same. The Mercedes Benz frame is sometimes referred as roots of cube of unity in $R^2(R)$. These vectors are nothing but simple representation of the cube roots of the unity in the 2-Dimnessional space.
The frame is known after the famous logo of Mercedes Benz

Similar geometric example existed in 3-Dimensional space, which is formed by the vertices of a triangular prism with centroid at origin.
Equiangular tight frames with $n + 1$ elements in an n −dimensional space are called regular simplices.
Equiangular frames are connected with combinatorial designs and are also connected to strongly regular graphs. These topics are of applied nature and equiangular frames are used to study these. Through equiangular frames, the structures of graphs are studied. One of the

consequences of these frames are the optimal packing of the graphs and the stability structure of the graphs for optimality packing. This direction of equiangular frames are not discussed in the thesis. Interested readers may go through the relevant references to develop and enhance their knowledge of uses of equiangular frames in graph theory and combinatory.
Various other directions of research related to applied aspects of equiangular frames can be found in the references.

Near Equiangular Frames
Now, we move to the new topic to near equiangular frames.
Generalizing orthonormal frames, we introduce almost orthonormal frames as
Definition 2. A frame $\{x_n\}_{n\in I}$ is said to be almost orthonormal frame if there exists a finite set $G \subseteq I$ such that$\{x_n\}_{n\in I\setminus G}$ is an orthonormal frame.
Almost orthogonal frames are recently introduced and studied. The applications of these frames are also discussed to connect these frames to the famous conjecture called Feichtinger conjecture.
The Feichtinger conjecture enquires about expressing every bounded frame as finite union of Riesz basic sequences. Interestingly, the conjecture is solved in affirmative answer very recently in 2016.
Generalizing equiangular frames, we introduce near equiangular frames as
Definition 3. A frame $\{x_n\}_{n\in I}$ is said to be near equiangular frame or almost equiangular frames if it satisfies$\langle x_i, x_j\rangle = k$, for all $i, j\ (i \neq j) \in I\setminus G$, where k is a constant and $G \subseteq I$ is a finite set.
Analysing the definition of near equiangular frames, one can observe that equiangular frames are near equiangular frames as G can be chosen $\emptyset$ set. The following example illustrate that near equiangular frames are generalizations of equiangular frames.
Example 2. Let $H = l^2(N)$ and let $\{e_n\}_{n\in N}$ be the standard/ideal orthonormal basis. Define a sequence $\{x_n\}_{n\in N}$ as

$$x_1 = \frac{e_1}{2}, x_2 = \frac{e_1}{2}, x_n = e_{n-2}; n \geq 3.$$

Then $\{x_n\}_{n\in N}$ is a frame for H and

$$\langle x_i, x_j\rangle = 0; i \neq j; i, j \geq 2$$

and

$$\langle x_1, x_2\rangle = \frac{1}{4}.$$

Then $\{x_n\}_{n\in \mathbb{N}}$ is not an equiangular frame but choosing $G = \{1\}$, it satisfies the condition of near equiangular frame.
Hence, it is a near equiangular frame which is not an equiangular frame.
Also, one may note that if a frame in infinite dimensional Hilbert space isnear equiangular frame, then using the frame inequality, it is easy to deduce that the constant k must be zero and hence near equiangular frame is almost orthogonal frame.
In case of finite dimensional Hilbert spaces, these two concepts can exist simultaneously.
One may consider a Mercedes Benz frame in $R^2(R)$ together with a suitable element of the space.
Similar examples may be constructed using the technique of Mercedes Benz frame. Examples in abstract spaces like $l^2(N)$ or $L^2(R)$ are bit tedious to work with and hence they are not considered much in the compilation of the thesis. However, the setup of near equiangular frames would work on the same lines without any serious theoretical exceptions. Therefore, we drive our attention to the results obtained regarding the near equiangular frames and try to visualize the results obtained through the illustrative examples.

Some Results

The very first result connects almost orthogonal frames with Riesz frames.

Theorem 1. Let $\{x_n\}_{n\in I}$ be an almost orthogonal frame. Then, $\{x_n\}_{n\in I}$ is containing a Riesz basis.

Proof. Let $\{x_n\}_{n\in I}$ be an almost orthogonal frame. It is to show there exists a finite set G which gives that $\{x_n\}_{n\in I\setminus G}$ is an orthogonal frame.

We claim that there is a sub-collection of $\{x_n\}_{n\in I}$, which is an exact frame or a Riesz basis.

Consider $\{x_n\}_{n\in G}$. Then $[x_n]_{n\in G}$ is a finite dimensional subspace of the space.

Since $\{x_n\}_{n\in G}$ is complete in the subspace and hence is a frame for $[x_n]_{n\in G}$. Also, it is clear that $\{x_n\}_{n\in G}$ is a near exact frame for $[x_n]_{n\in G}$. It can be made Riesz basis for $[x_n]_{n\in G}$ by dropping finite number of elements from the collection.

Then, let $\{x_{n'}\}_{n'\in G}$ be the extracted Riesz basis for $[x_n]_{n\in G}$.

Now consider $\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$. Clearly, $\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$ is a frame for the space. Also, note that the frame is exact frame for the space.

Hence $\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$ is a Riesz basis, which proves the claim.

Before proceeding further, let us consider an illustrative example given below for better understanding.

Example 3. Let $H = l^2(N)$ and let $\{e_n\}_{n\in N}$ be the orthonormal basis. Define a sequence $\{x_n\}_{n\in N}$ as

$$x_1 = x_2 = \frac{e_1}{2},\ x_3 = x_4 = x_5 = e_2,\ x_n = e_{n-3}; n \geq 6.$$

Then $\{x_n\}_{n\in N}$ is a near equiangular frame for H with $\langle x_i, x_j\rangle = 0; i \neq j; i,j \geq 6$. Then $\{x_n\}_{n\in N}$ contains an exact frame or Riesz basis namely $\left\{\frac{e_1}{2}, e_2, e_3, e_4, \dots, e_n, \dots\right\}$.

Similar other examples with slight variations can be worked out in similar fashion for illustrations.

Almost orthogonal frames are also bounded frames as stated below

Theorem 2. Let $\{x_n\}_{n\in I}$ be an almost orthogonal frame without any zero element. Then, $\{x_n\}_{n\in I}$ is a bounded frame.

Proof. Let $\{x_n\}_{n\in I}$ be an almost orthogonal frame. Then there is a finite set G such that $\{x_n\}_{n\in I\setminus G}$ is an orthogonal frame sequence.

We need to prove that $\{x_n\}_{n\in I}$ is a bounded frame for the space.

In order to show this, we show $\{x_n\}_{n\in I\setminus G}$ and $\{x_n\}_{n\in G}$ are bounded frame sequences and it will be proved.

First consider $\{x_n\}_{n\in G}$. It is a finite collection of non-zero elements and hence must be having an element x_i (say) with minimum norm i.e. $\|x_n\| \geq \|x_i\|; n \in G; i \in G$, where $i \in G$ is a specific index.

Now consider $\{x_n\}_{n\in I\setminus G}$. Contrary, let for any very small $\varepsilon > 0$, there exists x_{i_ε} such that $\|x_{n_\varepsilon}\| < \varepsilon$.

Note that $\langle x_{n_\varepsilon}, x_n\rangle = 0; n \neq n_e; n \in I$. Also, $\{x_n\}_{n\in I}$ is a frame for the space. There exist constants A and B with $0 < A, B < \infty$ satisfying the inequality

$$A\|x\|^2 \leq \sum_{n\in I}|\langle x, x_n\rangle|^2 \leq B\|x\|^2.$$

In general, without any loss, assume $A \geq \varepsilon^2$.

Choose $x = x_{n_\varepsilon}$. Then $\sum_{n\in I}\left|\langle x_{n_\varepsilon}, x_n\rangle\right|^2 = \left|\langle x_{n_\varepsilon}, x_{n_\varepsilon}\rangle\right|^2 = \left\|x_{n_\varepsilon}\right\|^4$.

Using frame inequality, we have $A\left\|x_{n_\varepsilon}\right\|^2 \leq \left\|x_{n_\varepsilon}\right\|^4$ or $A \leq \left\|x_{n_\varepsilon}\right\|^2 < \varepsilon^2$.

But this is a contradiction. Thus $\{x_n\}_{n\in I}$ is a bounded frame sequence.

Hence $\{x_n\}_{n\in I}$ is a bounded frame.

The following example explain the result obtained.

Example 4. Let $H = l^2(N)$ and let $\{e_n\}_{n\in N}$ be the ideal orthonormal basis. Define a sequence $\{x_n\}_{n\in N}$ as

$$x_n = x_{n+1} = e_1; n = 1, \qquad x_n = x_{n+1} = x_{n+2} = e_2; n = 3,$$
$$x_n = x_{n+1} = x_{n+2} = e_3; n = 6,\ x_n = e_{n-5}; n \geq 9.$$

Then $\{x_n\}_{n\in N}$ is a near equiangular frame for H with $\langle x_i, x_j\rangle = 0; i \neq j; i,j \geq 9$. Then $\{x_n\}_{n\in N}$ is a bounded frame. Further, if someone considers an unbounded sequence of the form $\left\{e_1, \frac{e_2}{2}, \frac{e_3}{3}, \frac{e_4}{4}, \dots, \frac{e_n}{n}, \dots\right\}$ or similar form, then it is almost orthogonal unbounded sequence, but in that case it is not a frame sequence for the space.

Other examples with slight variations can be considered to arrive at the conclusion of the theorem.

Similar results may be worked out for almost orthonormal frames with slight modifications.

The next result states that near equiangular frames can be expressed as finite union of Riesz basic sequences or Riesz frame sequences.

Theorem 3. Let $\{x_n\}_{n\in I}$ be a near equiangular frame. Then, $\{x_n\}_{n\in I}$ can be written as finite union of equiangular frame sequences.

Proof. Let $\{x_n\}_{n\in I}$ be a near equiangular frame. Then, we show that $\{x_n\}_{n\in I}$ is written as finite union of equiangular frame sequences. In other words, $\{x_n\}_{n\in I} = \cup_i \{x_{n_i}\}$, where each $\{x_{n_i}\}$ is an equiangular frame and index i' varies over a finite set.

Since $\{x_n\}_{n\in I}$ is a near equiangular frame, there exists a finite set G of I such that $\langle x_i, x_j\rangle = k; i \neq j;\ i,j \in I\backslash G$.

Consider $\{x_n\}_{n\in G}$. Since G is a finite set, then the collection $\{x_n\}_{n\in G}$ can be written as finite union of singleton and each singleton collection is an equiangular frame sequence.

If $\{x_n\}_{n\in I}$ is a near equiangular frame for infinite dimensional Hilbert space, therefore as discussed earlier the constant k must be zero and hence the frame is almost orthogonal frame. Using Theorem 1, one can prove that it may be written as finite union of singletons and one orthogonal collection.

Each sub-collection is clearly an equiangular frame sequence.

Hence, the frame can be expressed as finite union of equiangular frame sequences and the proof is completed.

If $\{x_n\}_{n\in I}$ is a near equiangular frame for finite dimensional Hilbert space, then the index n varies over a finite range (i.e. I is finite) and it can be expressed as finite union of singletons each of which is considered as an equiangular frame.

If the index varies over an infinite range (i.e. I is infinite), then with slight manipulation using a suitable element, it can be shown that $\{x_n\}_{n\in I}$ does not satisfy the frame inequality.

This is a contradiction to the given hypotheses.

Hence, the frame $\{x_n\}_{n\in I}$ can be written as finite union of equiangular frames.

Example 5. Let $H = l^2(N)$ and let $\{e_n\}_{n\in \mathbb{N}}$ be the orthonormal basis. Define a sequence $\{x_n\}_{n\in N}$ as

$$x_n = x_{n+1} = \frac{e_1}{2}; n = 1, \qquad x_n = x_{n+1} = x_{n+2} = \frac{e_2}{3}; n = 3,$$
$$x_n = x_{n+1} = x_{n+2} = x_{n+3} = \frac{e_3}{4}; n = 6\ ,\ x_n = e_{n-6}; n \geq 10.$$

Then $\{x_n\}_{n\in N}$ is a near equiangular frame for H with $\langle x_i, x_j\rangle = 0; i \neq j; i,j \geq 10$. Then $\{x_n\}_{n\in N}$ is a bounded frame. Clearly, $\{x_1, x_3, x_6, x_{10}, \dots, x_n, \dots\}$ is an equiangular frame sequence. Similarly, $\{x_2, x_4, x_7\}$ is an equiangular frame sequence and likewise, $\{x_5, x_8\}$ and $\{x_9\}$ are equiangular frame sequences. Thus

$$\{x_n\}_{n\in I} = \{x_1, x_3, x_6, x_{10}, \dots, x_n, \dots\} \cup \{x_2, x_4, x_7\} \cup \{x_5, x_8\} \cup \{x_9\}.$$

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The near equiangular frame is written as finite union of equiangular frame sequences.
In many cases, a frame may not contain a Riesz frame. But the structure of equiangular frames and near equiangular frames is much refined and every such frame contains a Riesz frame as stated in the next result. It precisely indicates that near equiangular frames are more like bases than any other arbitrary frame.
Theorem 4. Let $\{x_n\}_{n\in I}$ be a near equiangular frame. Then, $\{x_n\}_{n\in I}$ contains a Riesz frame.
Proof. Let $\{x_n\}_{n\in I}$ be a near equiangular frame. We claim that there is a sub-collection of$\{x_n\}_{n\in I}$, which is an exact frame or a Riesz basis.
Consider$\{x_n\}_{n\in G}$. Then $[x_n]_{n\in G}$ is a finite dimensional subspace of the space.
Since$\{x_n\}_{n\in G}$ is complete in the subspace and hence is a frame for$[x_n]_{n\in G}$. Also, it is clear that$\{x_n\}_{n\in G}$ is a near exact frame for$[x_n]_{n\in G}$. It can be made Riesz basis for$[x_n]_{n\in G}$ by dropping finite number of elements from the collection.
Then, let$\{x_{n'}\}_{n'\in G}$ be the extracted Riesz basis for$[x_n]_{n\in G}$.
If$\{x_n\}_{n\in I\setminus G}$ is a near equiangular frame for infinite dimensional Hilbert space, then as discussed earlier the constant k must be zero and hence the frame is almost orthogonal frame.
Using Theorem 1, one can prove that a Riesz basis sequence may be extracted from the collection. Each sub-collection is clearly an equiangular frame sequence and collectively, elements of these sub-sequences span the whole Hilbert space.
Hence, the frame contains a Riesz basis for the Hilbert space and the proof is completed.
If$\{x_n\}_{n\in I}$ is a near equiangular frame for finite dimensional Hilbert space, then again, we can extract a Riesz basis for the finite dimensional space from the collection.
Now consider$\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$. Clearly,$\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$ is a frame for the space.
Also, note that the frame is exact frame for the space.
Hence$\{x_{n'}\}_{n'\in G} \cup \{x_n\}_{n\in I\setminus G}$ is a Riesz basis, which proves the claim.
If the index varies over an infinite range (i.e. I is infinite), then with slight manipulation using a suitable element, it can be shown that$\{x_n\}_{n\in I}$ does not satisfy the frame inequality. This is a contradiction to the given hypotheses.
Working out all possibilities, we can deduce that we are able to extract a Riez basis from the given near equiangular frame.
This completes the proof.
The example provided below illustrate the result in a more simple way for the visualization and understanding.
Example 6. Let $H = l^2(N)$ and let $\{e_n\}_{n\in N}$ be the orthonormal basis. Define a sequence $\{x_n\}_{n\in N}$ as

$$x_n = x_{n+1} = e_1; n = 1, \qquad x_n = x_{n+1} = x_{n+2} = e_2; n = 3,$$
$$x_n = x_{n+1} = x_{n+2} = x_{n+3} = e_3; n = 6\ ,\ x_n = e_{n-6}; n \geq 10.$$

Then $\{x_n\}_{n\in N}$ is a near equiangular frame for H with$\langle x_i, x_j\rangle = 0; i \neq j; i,j \geq 10$. Then $\{x_n\}_{n\in N}$ is a bounded frame. Clearly,$\{x_1, x_3, x_6, x_{10}, \dots, x_n, \dots\}$ is an equiangular frame sequence. Also$[x_1, x_3, x_6, x_{10}, \dots, x_n, \dots] = H$. Moreover, it is an exact frame sequence. Thus$\{x_1, x_3, x_6, x_{10}, \dots, x_n, \dots\}$ is a Riesz basis for H, which is contained in the given near equiangular frame.
Bounded frames are again very important class of frames which are widely used in many applications.
The following result discuss one of the consequences of near equiangular frames in infinite dimensional Hilbert spaces connecting these to bounded frames.

Theorem 5. Let $\{x_n\}_{n\in I}$ be a near equiangular frame with $\langle x_i, x_j\rangle = k$, for all $i, j\ (i \neq j) \in I\backslash G$, where k is a constant and $G \subseteq I$ is a finite set, I is an infinite set and $[x_n]_{n\in I} = H$. If $\{x_n\}_{n\in I}$ is a bounded frame, then $k = 0$.

Proof. Let $\{x_n\}_{n\in I}$ be a near equiangular frame with $\langle x_i, x_j\rangle = k$, for all $i, j\ (i \neq j) \in I\backslash G$, where k is a constant and $G \subseteq I$ is a finite set and $[x_n]_{n\in I} = H$. Let $\{x_n\}_{n\in I}$ be a bounded frame. Then, it is to show that $k = 0$. On the contrary, let assume $k \neq 0$.

Without any loss of generality, let $k > 0$.

Since $\{x_n\}_{n\in I}$ is a near equiangular frame, then $\langle x_i, x_j\rangle = k; i \neq j; i, j \in I\backslash G$.

Let $x = x_m$, where m is the smallest number in $I\backslash G$.

Consider $\sum_{i\in I\backslash G;\mathrm{i}\neq\mathrm{m}} |\langle x, x_i\rangle|^2$. Then

$$\begin{aligned}\sum_{i\in I}|\langle x, x_i\rangle|^2 &= |\langle x, x_1\rangle|^2 + |\langle x, x_2\rangle|^2 + |\langle x, x_3\rangle|^2 + \cdots \\ &\quad + |\langle x, x_m\rangle|^2 + \cdots \\ &\geq |\langle x, x_m\rangle|^2 + |\langle x, x_{m+1}\rangle|^2 \ldots \\ &\geq k^2 + k^2 \ldots.\end{aligned}$$

Thus, the sum $\sum_{i\in I}|\langle x, x_i\rangle|^2$ is not bounded as $k > 0$ and hence it cannot be less than $B\|x\|^2$, as required to satisfy the frame inequality.

Therefore, $\{x_n\}_{n\in I}$ is not a frame, which is a contradiction to the given hypotheses.

So, assumption $k \neq 0$ is wrong.

This completes the proof.

Corollary 1. Let $\{x_n\}_{n\in I}$ be a near equiangular frame with $\langle x_i, x_j\rangle = k$, for all $i, j\ (i \neq j) \in I\backslash G$, where k is a constant and $G \subseteq I$ is a finite set, I is an infinite set and $[x_n]_{n\in I} = H$. If $k \neq 0$, then $\{x_n\}_{n\in I}$ is not a bounded frame.

The above result provide the connection of equiangular frames with bounded frames in any infinite dimensional space.

Regarding illustrations with examples, one may consider any of the above examples or similar examples.

Conclusion

In the present paper , we highlighted an important and a vital aspect of the theory of frames. In the paper , equiangular frames are discussed and studied. Generalizing these frames, a new concept of near equiangular frames is introduced and studied. These frames are similar to equiangular frames or orthogonal frames but the conditions imposed on the structure of frames are less compared with former two types of frames.

The paper summary may be expressed as follows

The concepts of orthonormal frames and equiangular frames are generalized Hilbert spaces. Almost orthonormal frames and near equiangular frames in Hilbert spaces are studied, discussed and few results are obtained in this direction. Various consequences of these generalizations are also discussed. Wherever the need of some illustrative examples arises, the same are provided for better understanding of the concepts. Applications and advantages of these generalizations in applications is yet to explore, which may be pursued in the subsequent work.

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